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a use of non-convergent series to obtain it (*e. g.*, for $|k| \geqq 1$ in the above). This method of finding important identities (2) has been used freely by Hermite and many others without question of its validity, and without offering independent proofs of (2).

2673. Proposed by WILLIAM O. BEAL, University of Minnesota.

A plane through the center of an oblate spheroid makes an angle, i, with the plane of its equator. Express the eccentricity, e', of this section in terms of the eccentricity, e, of a meridian section and the angle, i.

2674. Proposed by J. O. MAHONEY, Forest Avenue High School, Dallas, Texas.

If two sides of a triangle differ by less than a certain length, e, the two opposite angles will differ by less than a certain quantity λ, expressed in degrees, such that $\lambda < 61e/a$, where a expresses, with a possible error e, the length of the apparently equal sides of the triangle.

2675. Proposed by E. B. ESCOTT, Kansas City, Mo.

Sum the series

$$-\frac{1}{2}\cdot\frac{1^3x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{2^3x^5}{5}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{3^3x^7}{7}+\cdots.$$

2676. Proposed by EDWIN R. SMITH, State College, Pa.

Find the maximum term of the series

$$\frac{sp(sp-1)\cdots(sp-r+1)}{s(s-1)\cdots(s-r+1)}F(-r,\,-sq,\,sp-r+1,\,1)$$

where s, r, sq and sp are positive integers, $r < s$, p and q are proper fractions such that $p + q = 1$, and $F(-r,\,-qs,\,sp-r+1,\,1)$ is a hypergeometric series. If s, r and $s - r$ are large, show that the maximum term is approximately equal to

$$\sqrt{\frac{s}{2\pi pqr(s-r)}}.$$

2677. Proposed by R. K. MORLEY, Worcester, Mass.

A quarter-mile track is to be constructed, having semi-circular ends and straightaway sides. It is required to have the rectangular part of enclosed field referred to in No. 12, Granville's Calculus, page 116, as large as possible. Find length of the straightaways.

Also, required to inscribe the maximum rectangle in a track of length l, with semi-circular ends and straightaway sides, assuming that two sides of the rectangle are parallel to the straightaways. Find the length of the straightaways and the dimensions of the rectangle.

2678. Proposed by C. F. GUMMER, Queen's University, Kingston, Canada.

Find necessary and sufficient conditions that the roots of the equation

$$x^{n+1}+a_1x^n+a_2x^{n-1}+\cdots+a_{n+1}=0$$

may be all real and separated by the roots of $x^n+b_1x^{n-1}+b_2x^{n-2}+\cdots+b_n=0$.

2679. Proposed by J. W. LASLEY, University of North Carolina.

Show that the perpendicular from any point on a circle to any chord of the circle is a mean proportional to the perpendiculars from that point to the tangents at the ends of the chord.

SOLUTIONS OF PROBLEMS.

Note.—Prepare solutions as follows: (1) Give the number of the problem with the name of the proposer and his address; (2) restate the problem as published; (3) then give the name of the solver with address; (4) then give the solution carefully and neatly written.

Solutions of 477 (Algebra) were received from E. B. Escott and E. H. Worthington; of 478 (Algebra) from E. B. Escott; and of 481 (Algebra) from L. C. Mathewson. These solutions were received after copy had been prepared for publication. Professor Worthington should have received credit also for solving 476 (Algebra).

484 (Algebra). Proposed by E. V. HUNTINGTON, Cambridge, Mass.

Show that

$$\frac{\dfrac{1}{m^2}-\dfrac{k_1}{(m+1)^2}+\dfrac{k_2}{(m+2)^2}-\cdots+\dfrac{(-1)^k}{(m+k)^2}}{\dfrac{1}{m}-\dfrac{k_1}{m+1}+\dfrac{k_2}{m+2}-\cdots+\dfrac{(-1)^k}{m+k}}=\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{m+k}$$

for all positive integral values of m and k. Here

$$k_1=\frac{k}{1},\quad k_2=\frac{k(k-1)}{1\cdot 2},\quad k_3=\frac{k(k-1)(k-2)}{1\cdot 2\cdot 3},\quad \text{etc.}$$

This equation was suggested to the proposer by a professor of chemistry who wishes to make use of the equation, if correct, in an actual problem in bacteriology.

I. Solution by David F. Barrow, Sheffield Scientific School.

The denominator may be summed as follows: Consider

$$\int_0^1 x^{m-1}(1-x)^k dx.$$

If we expand the integrand by the binomial theorem and integrate term by term, we obtain the denominator. But if we integrate by parts, taking $u=(1-x)^k$ and $dv=x^{m-1}dx$, we obtain

$$\left.\frac{x^m(1-x)^k}{m}\right]_0^1+\frac{k}{m}\int_0^1 x^m(1-x)^{k-1}dx.$$

The closed part vanishes at both limits. This furnishes a convenient reduction formula by successive applications of which the integration may be completed without difficulty. We are thus led to the result,

$$\frac{1}{m}-\frac{k_1}{m+1}+\frac{k_2}{m+2}-\cdots+\frac{(-1)^k}{m+k}=\frac{k!(m-1)!}{(m+k)!}.$$

In the formula to be proved, let us multiply up this denominator on the right side.

$$(1)\quad \frac{1}{m^2}-\frac{k_1}{(m+1)^2}+\frac{k_2}{(m+2)^2}-\cdots+\frac{(-1)^k}{(m+k)^2}=\frac{k!(m-1)!}{(m+k)!}\left(\frac{1}{m}+\frac{1}{m+1}+\cdots+\frac{1}{m+k}\right).$$

Now if $k=0$ or if $k=1$, this formula is easily seen to be true for all positive integral values of m. We shall complete the proof by mathematical induction if we establish that *if* (1) *is true for all values of m and any particular value of k, then it is true for all values of m and that value of k increased by unity.*

We assume, then, that (1) is an identity in m when $k=k$. We may therefore replace m by $m+1$, obtaining

$$(2)\quad \begin{aligned}\frac{1}{(m+1)^2}-\frac{k_1}{(m+2)^2}+\frac{k_2}{(m+3)^2}-\cdots+\frac{(-1)^k}{(m+k+1)^2}\\ =\frac{k!m!}{(m+k+1)!}\left(\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{m+k+1}\right).\end{aligned}$$

Now $k_1, k_2, \cdots$ are the binomial coefficients. Hence they satisfy the relation $k_{i-1}+k_i=(k+1)_i$ (preserving the notation of the proposer). Remembering this, we subtract equation (2) from equation (1), and get

$$\frac{1}{m^2}-\frac{(k+1)_1}{(m+1)^2}+\frac{(k+1)_2}{(m+2)^2}-\cdots+\frac{(-1)^{k+1}}{(m+k+1)^2}$$
$$=\frac{k!(m-1)!}{(m+k)!}\left[\frac{1}{m}+\left(1-\frac{m}{m+k+1}\right)\left(\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{m+k}\right)\right.$$
$$\left.-\frac{m}{m+k+1}\cdot\frac{1}{m+k+1}\right]$$
$$=\frac{k!(m-1)!}{(m+k)!}\left[\frac{(m+k+1)^2-m^2}{m(m+k+1)^2}+\frac{k+1}{m+k+1}\left(\frac{1}{m+1}+\cdots+\frac{1}{m+k}\right)\right]$$
$$=\frac{(k+1)!(m-1)!}{(m+k+1)!}\left(\frac{1}{m}+\frac{1}{m+1}+\cdots+\frac{1}{m+k+1}\right).$$

But this is what we get if in (1) we replace k by $k+1$. Hence (1) is true for all values of m and for $k+1$; hence it is true for all values of m and k.

II. Solution by Edwin Bidwell Wilson, Massachusetts Institute of Technology.

The equation is an identity for all values of m and is merely a special case of a general theorem in logarithmic differentiation of rational fractions.

Suppose that

$$A\,\frac{(x-a_1)(x-a_2)\cdots(x-a_m)}{(x-b_1)(x-b_2)\cdots(x-b_n)}$$

is a rational fraction, and for simplicity let it be proper ($m<n$) and without multiple factors in either numerator or denominator (the result for the general rational fraction is more complicated only in notation).

The derivative of the logarithm of this fraction is

$$\frac{1}{x-a_1}+\frac{1}{x-a_2}+\cdots+\frac{1}{x-a_m}-\frac{1}{x-b_1}-\frac{1}{x-b_2}-\cdots-\frac{1}{x-b_n}.$$

The fraction may, however, be resolved into partial fractions as

$$\frac{c_1}{x-b_1}+\frac{c_2}{x-b_2}+\cdots+\frac{c_n}{x-b_n}$$

and its logarithmic derivative then has the form

$$-\frac{\dfrac{c_1}{(x-b_1)^2}+\dfrac{c_2}{(x-b_2)^2}+\cdots+\dfrac{c_n}{(x-b_n)^2}}{\dfrac{c_1}{x-b_1}+\dfrac{c_2}{x-b_2}+\cdots+\dfrac{c_n}{x-b_n}},$$

which must be identical with the former expression.

In the particular case mentioned in the problem the rational fraction is evidently of the form

$$\frac{A}{m(m+1)(m+2)\cdots(m+k)},$$

m replacing x. The expansion into partial fractions is

$$\frac{k_0}{m}+\frac{k_1}{m+1}+\frac{k_2}{m+2}+\cdots+\frac{k_k}{m+k}$$

and the coefficients k_p may readily be found by the "substitution" method. Indeed, if we clear of fractions and set $m=-p$ we have

$$A=k_p(-p)(-p+1)\cdots(-1)(1)(2)\cdots(k-p)$$

or

$$k_p=\frac{(-1)^pA}{p!(k-p)!}=(-1)^p\frac{k(k-1)\cdots(k-p+1)}{1\cdot2\cdots p}$$

if the as yet undetermined A be taken as $k!$.

Another interesting special case is found on treating

$$\frac{A}{(m-k)(m-k+1)\cdots(m-1)m(m+1)\cdots(m+k-1)(m+k)},$$

especially if k be allowed to become infinite (cotangent series).

Also solved by FRANK IRWIN, C. F. GUMMER, S. BEALTY, O. SCHMIEDEL, OLIVE C. HAZLETT, and RALPH KEFFER.

485 (Algebra). Proposed by J. L. WALSH, Madison, Wisconsin.

Is it true that to every convergent series of positive terms, $a_1 + a_2 + a_3 + \cdots$, there corresponds a series of the type

$$\frac{M}{1^p} + \frac{M}{2^p} + \frac{M}{3^p} + \cdots,$$

such that $M/k^p > a_k$, $p > 1$?

SOLUTION BY E. H. MOORE, The University of Chicago.

A convergent series of positive terms, $a_1 + a_2 + \cdots + a_k + \cdots$, is a positive-valued function α; $\alpha(k) = a_k$ ($k = 1, 2, 3, \cdots$), of the variable positive integer k, satisfying the condition that the corresponding series is convergent. Denote the class of all such functions α by $\mathfrak{C}_+$.

If the question proposed is answered in the affirmative, then there exists a sequence α_n ($n = 1, 2, 3, \cdots$) of functions α_n of the class $\mathfrak{C}_+$, viz., the sequence of functions

$$\alpha_n : \alpha_n(k) = \frac{n}{k^{1+(1/n)}} \quad (k = 1, 2, 3, \cdots),$$

of such a nature that every function α of the class $\mathfrak{C}_+$ is dominated by a suitably chosen function α_n of the sequence, viz., $|\alpha(k)| \leqq |\alpha_n(k)|$ ($k = 1, 2, 3, \cdots$),—that is, the class $\mathfrak{C}_+$ has the dominance property D_2 defined (for the general class of real-valued functions on the general range) in § 22 of my *Introduction to a Form of General Analysis* (The New Haven Mathematical Colloquium, Yale University Press, 1910, p. 41).

Now I have proved (§ 23*c* (5), *loc. cit.*) that the class $\mathfrak{M}^{\mathrm{III}_1}$ (and, a fortiori, the class $\mathfrak{C}_+$) of all absolutely convergent series of real-valued terms fails to have the dominance property D_2. Hence, the question proposed must be answered in the negative.

The proposition cited is one of a number of theorems involving various dominance properties. The present question may be answered still more luminously by citing the theorem of Hadamard (Acta Mathematatica, vol. 18, 1894, p. 328, theorem (β); cf. also *loc. cit.*, p. 49) that for every sequence (α_n) of functions of the class $\mathfrak{C}_+$ such that for every k $\alpha_n(k)$ increases with n there exists a function α of the class $\mathfrak{C}_+$ of such a nature that for every n

$$\lim_{k=\infty} \frac{\alpha_n(k)}{\alpha(k)} = 0.$$

Also solved by ELIJAH SWIFT.

486 (Algebra). Proposed by FLORENCE P. LEWIS, Baltimore, Md.

Find the condition which must be satisfied by the coefficients of the quartic

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$

in order that the equation be solvable by successive applications of the quadratic formula.

SOLUTION BY THE PROPOSER.

The equation must be of the form $A(ax^2 + bx + c)^2 + B(ax^2 + bx + c) + C = 0$ or $Ay^2 + By + C = 0$. Let x_1 and x_1' correspond to root y_1, and x_2 and x_2' to root y_2, respectively. We then have $x_1 + x_1' = -b/a$ and $x_2 + x_2' = -b/a$. This says that the line joining x_1 and x_1' is bisected at $-b/2a$, or that x_1 and x_1' are harmonic conjugates as to $-b/2a$ and ∞. The same